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Research Article

Distributed H_∞ Consensus Filtering for Parabolic Distributed Parameter Systems with Multiple Missing Measurements: A Mobile Sensing Approach

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This paper investigates the distributed H_∞ consensus filtering issue for a class of distributed parameter systems with bounded disturbance. In a framework of optimizing performance, a new approach to improving filter performance is proposed by employing mobile sensor networks. Moreover, the information missing in mobile sensor networks is modeled as a conditional probability distribution. The aim of the filtering challenge is to construct a distributed consensus filter such that the filtering error system is globally asymptotically stable in the mean square, and what disturbances do to the estimation accuracy is attenuated at the H_∞ consensus performance level. Utilizing the Lyapunov direct approach and the spatial operator technique, several sufficient criteria are given for the proposed filter to satisfy the H_∞ consensus performance constraint. Finally, a numerical simulation is given to demonstrate the effectiveness of the design scheme of the proposed filter.

1. Introduction

Over the last decade or so, wireless sensor networks (WSNs) have been successfully applied in a wide range of areas, such as environmental monitoring, military applications, smart buildings, and health management [1]. A sensor network, in general, comprises a group of sensing devices that can communicate wirelessly to coordinate their task and collect data. Sensor network observations are required to be processed in order to ensure decision-making. As a result, the problem of state estimation for a wide variety of systems has attracted considerable attention. A centralized data processing method can be used to estimate the state of a system, determine its parameters, and identify its detection sources. To study the state estimation problem of discrete linear systems with Markovian delay and packet losses, a class of limited received history estimators based on the idea of jump linear estimators was proposed in [2].

Yet each sensing node is equipped with an embedded device which has a limited power. The use of distributed estimation is an effective way to save energy. A sensor scheduling strategy was proposed in [3] based on energy conservation while focusing on distributed state estimation for wireless sensor networks under energy constraints. In [4], a robust performance-preserving state estimator was designed to obtain state estimation of discrete-time complex networks with estimation errors satisfying mean-square exponential convergence for real networks with packet losses and noise interference in data transmission. To further extend the estimation accuracy in case of data missing, H_∞ estimation is considered. An H_∞ estimator was designed in [5], which achieved the state estimation of a class of time-varying neural networks under measurement degradation and randomly occurring deception attacks, such that the predefined probabilistic constraints of error dynamics were satisfied and that H_∞ performance was guaranteed. It is noted that the systems to which these state estimates are applied are lumped parameter systems.

Also, the estimation of distributed parameter systems is receiving more attention recently. The work to predict the water quality concentration distribution in the water supply appeared in [6], where the system was modeled by the reaction diffusion equation. For distributed parameter processes [7], optimal state estimators are provided based on optimal sensor locations. An adaptive consensus filter was proposed in [8] for filtering in distributed parameter systems with fixed sensors located in the system in a spatially distributed way.

Data collected by a network of mobile sensors are another type of estimate in spatially distributed systems. In [9], an output feedback controller was designed using a mobile sensor network with actuators to stabilize a class of randomly distributed parameter systems faster. With the introduction of this mobile actuator-sensor network, the control performance of the spatially distributed process is enhanced, and thus, an optimization framework for the control problem of distributed parameter systems is built. In [10], the estimation problem of spatially distributed processes described by a partial differential equation was studied using a group of sensors which could move in the spatial domain in order to enhance the performance of the state estimator. Compared to a group of immobile sensors, mobile sensors have the flexibility to move around when collecting data. Therefore, the number of sensors in the system can be reduced, which reduces energy consumption and increases efficiency. The state estimator designed with the guidance of mobile sensors is capable of achieving better estimating performance as shown in [10]. Related research work has been further extended to other aspects. In [11], an accuracy reconstruction of the traffic flow was achieved using data collected by a mobile sensor network for the Lighthill–Whitham–Richards model described by partial differential equations. And the parameter estimation of distributed processes by optimizing sensor locations was studied in [12].

Unfortunately, it is often the case that information regarding the system states is only partially available in practice. In real-world engineering, sensor temporal failure or network transmission delay are the main causes of the missing measurement phenomenon in networked environments. The first study addressing the filtering problem with missing measurements defined by Bernoulli distributions appeared in [13], and works addressing other lumped parameter systems with probabilistic missing data followed in [14–16]. The studies mentioned above brought up the fact that the observed output from fixed sensors was incomplete. In the measured output produced by mobile sensor networks, missing data have, however, often been ignored up to this point. A potential challenge of this issue is that the coefficients of the evolution equation which represent the spatial distribution process are operators and need to be dealt with by functional analysis.

As was previously stated, few studies have been undertaken to deal with distributed H_∞ consensus filtering systems with missing measurements. Our motivation for writing this paper is to focus on the problem of distributed H_∞ consensus filtering for parabolic distributed parameter systems with multiple missing measurements utilizing mobile sensor networks. Our approach aims to derive the state of spatially dispersed processes from the output with missing data in such a way that the filtering error reaches zero asymptotically stable in the mean square and that a certain H_∞ perturbation rejection attenuation level is ensured. For each moving sensing device, optimal trajectory planning can be produced with the use of spatial operators and the Lyapunov direct method. After which, a distributed filter subject to H_∞ consensus performance is constructed while taking into account the use of mobile sensors to improve the filter's performance. To demonstrate the effectiveness on the proposed conditions, the simulation is offered as an example.

2. Problem Formulation

In chemical reactors, parabolic partial differential equations can often be used to model changes in concentration in chemical reactions and the evolution of temperature in heat exchange [17]. The spatial distribution of temperature in a chemical reactor can be described by the following parabolic partial differential equation:

$$\begin{cases} \dfrac{\partial L(t,\eta)}{\partial t} = \dfrac{\partial}{\partial \eta}\left(a(\eta)\dfrac{\partial L(t,\eta)}{\partial \eta}\right) - \varphi(L(t,\eta))L(t,\eta) + d(\eta)w(t), \\ \\ z(t) = \displaystyle\int_0^l b(\eta)L(t,\eta)\mathrm{d}\eta, \end{cases} \tag{1}$$

subject to the initial condition

$$L(0,\eta) = L_0(\eta), \tag{2}$$

and having the Dirichlet boundary condition as

$$L(t,0) = L(t,l) = 0,\ t \geq 0, \tag{3}$$

where $L(t,\eta)$ indicates that the system is in the state of the location η at the time t and η is a spatial variable, varying in $\Omega = [0,l]$. The other variable of the state is the time $t \in [0,+\infty)$. The diffusion coefficient $a(\eta) \geq a_0 > 0$. $\varphi(L(t,\eta))$ is a nonlinear function and satisfies $\varphi_m \leq \varphi(L(t,\eta)) \leq \varphi_M$, where φ_m and φ_M are known bounds. $w(t)$ is an external disturbance, and $d(\eta)$

denotes its spatial distribution. $z(t)$ is the output to be estimated, and $b(\eta)$ denotes the output's spatial distribution.

The spatial measurements from n mobile sensors are as follows:

$$y(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_n(t) \end{bmatrix} = \begin{bmatrix} \beta_1(t)\int_0^l c_1(\eta;\eta_1^s(t))L(t,\eta)\mathrm{d}\eta \\ \beta_2(t)\int_0^l c_2(\eta;\eta_2^s(t))L(t,\eta)\mathrm{d}\eta \\ \vdots \\ \beta_n(t)\int_0^l c_n(\eta;\eta_n^s(t))L(t,\eta)\mathrm{d}\eta \end{bmatrix}, \quad (4)$$

or

$$y_i(t) = \beta_i(t)\int_0^l c_i(\eta;\eta_i^s(t))L(t,\eta)\mathrm{d}\eta, i = 1,2,\ldots,n, \quad (5)$$

where $y_i(t)$ is the measurement result that the ith mobile sensor collected. The spatial distribution of the ith moving sensing device is given by a nonnegative bounded function $c_i(\eta;\eta_i^s(t))$, where $\eta_i^s(t)$ is the position function of the ith sensor with respect to the time t, varying in $[0,l]$. It should be noticed that $c_i(\eta;\eta_i^s(t))$ indicates that each sensor may have a different distribution.

The stochastic variable $\beta_i(t) \in \mathbf{R}, i = 1,2,\ldots,n$ has values 1 and 0 for every i with

$$\mathrm{Prob}\{\beta_i(t) = 1\} = \overline{\beta}_i, \text{ and } \mathrm{Prob}\{\beta_i(t) = 0\} = 1 - \overline{\beta}_i, \quad (6)$$

where $\overline{\beta}_i, i = 1,2,\ldots,n$ are given constants taking values in [0, 1].

It is assumed that $\beta_i(t)$ is independent of the system's initial state and is also independent of $w(t)$. Hence, it is easy to draw the following results:

$$\mathbb{E}\{\beta_i(t) - \overline{\beta}_i\} = 0 \text{ as well as } \mathbb{E}\left\{\left(\beta_i(t) - \overline{\beta}_i\right)^2\right\} = \overline{\beta}_i\left(1 - \overline{\beta}_i\right). \quad (7)$$

Remark 1. The distributed parameter control system in this paper receives the measurement output from the mobile sensor i. Probabilistic missing data are unavoidable in sensor networks as sensing nodes can move around, and there is only so much channel bandwidth for signal transmission. The new sensor network model includes missing data at random and can more accurately depict mobile sensor network reality. Note that such a data missing mode was introduced in [13] first by Nahi.

As a general matter, the spatial distribution of mobile sensors at each time-varying position $\eta_i^s(t)$ can be expressed as

$$c_i(\eta;\eta_i^s) = \begin{cases} c_i(\eta), & \text{if } \eta \in [\eta_i^s - \varepsilon_i, \eta_i^s + \varepsilon_i], \\ 0, & \text{otherwise}, \end{cases} \quad (8)$$

or

$$c_i(\eta;\eta_i^s(t)) = c_i(\eta)[H(\eta - (\eta_i^s - \varepsilon_i)) - H(\eta - (\eta_i^s + \varepsilon_i))], \quad (9)$$

which are depicted as two different heaviside step functions.

Remark 2. Nowadays, most of the spatial distributions [8] about mobile agents are described by the following expression:

$$c(\eta;\eta_i^s) = \begin{cases} 1, & \text{if } \eta \in [\eta_i^s - \varepsilon, \eta_i^s + \varepsilon], \\ 0, & \text{otherwise}. \end{cases} \quad (10)$$

The aforementioned expression of the assumption indicates that the network of sensors is homogeneous. And the distribution of sensors given in (8) shows that each moving sensing device in the mobile sensor network may have a different location distribution; that is, $c_i(\eta;\eta_i^s(t))$ denotes a nonhomogeneous network. Thus, since Assumption (8) is employed, the result of this study would be less conservative.

More generally, the distribution of each sensor also can be piecewise smooth in local. For instance, one sensor's distribution in a symmetric interval $[\eta_i^s - \varepsilon, \eta_i^s + \varepsilon]$ can be shown as

$$c_i(\eta) = \begin{cases} \dfrac{1}{\varepsilon}, & \text{if } \eta \in \left[\eta_i^s - \dfrac{\varepsilon}{3}, \eta_i^s + \dfrac{\varepsilon}{3}\right], \\ \dfrac{1}{2\varepsilon}, & \text{if } \eta \in \left[\eta_i^s - \dfrac{\varepsilon}{2}, \eta_i^s - \dfrac{\varepsilon}{3}\right) \cup \left[\eta_i^s + \dfrac{\varepsilon}{3}, \eta_i^s + \dfrac{\varepsilon}{2}\right), \\ 0, & \text{if } \eta \in \left[\eta_i^s - \varepsilon, \eta_i^s - \dfrac{\varepsilon}{2}\right) \cup \left[\eta_i^s + \dfrac{\varepsilon}{2}, \eta_i^s + \varepsilon\right). \end{cases} \quad (11)$$

Therefore, the more general distribution of each mobile sensor can be taken into account within the proposed framework as follows:

$$c_i(\eta;\eta_i^s) = \sum_{j=1}^{n} c_{ij}(\eta)[H(\eta - (\eta_{i0}^s + (j-1)\Delta h)) - H(\eta - (\eta_{i0}^s + j\Delta h))], \quad (12)$$

where $\Delta h = (\varepsilon_+ + \varepsilon_-)/m2$, $\eta_{i0}^s = \eta_i^s - \varepsilon^-, i, j = 1,2,\ldots,n$.

It is advantageous to rewrite the parabolic system (1) in an abstract way in order to apply the Lyapunov direct method for H_∞ performance analysis of the filtering error system and the optimal moving strategy of mobile sensors.

Let $\mathscr{H}$ be a Hilbert space that has the inner product $\langle\cdot,\cdot\rangle$, and its induced norm is $|\cdot|$. Consider that $\mathscr{B}$ is a reflective Banach space, which is continuous and densely embedded in $\mathscr{H}$, $\|\cdot\|$ as a norm of $\mathscr{B}$. $\mathscr{B}^*$ is $\mathscr{B}$'s conjugated dual, whereas $\|\cdot\|_*$ is its induced norm. It follows $\mathscr{B} \longrightarrow \mathscr{H} \longrightarrow \mathscr{B}^*$ with both continuously and embedding dense, and as a result, we have $|h| \le \alpha\|h\|, h \in \mathscr{B}$, where α [18] is positive constant.

Here, the linear operator $\mathscr{A}\colon \mathscr{B} \longrightarrow \mathscr{B}^*$ which is called state operator satisfying the following assumptions [18]:

(A1) $|\langle h, \mathscr{A}g\rangle| \leq \varrho_0\|h\|\|g\|$, $h, g \in \mathscr{B}$, where $\varrho_0 > 0$, namely, $\mathscr{A}$ is bounded.

(A2) $\langle h, -\mathscr{A}h\rangle \geq a_0\|h\|^2$, $h \in \mathscr{B}$, where $a_0 > 0$, namely, $-\mathscr{A}$ is coercive.

Aside from that, the perturbation operator $\mathscr{D}\colon \mathbf{R} \longrightarrow \mathscr{B}^*$ can be defined as

$$\langle \mathscr{D}w, h\rangle = \int_0^l d(\eta)w(t)h(\eta)\mathrm{d}\eta, \quad \text{or,} \quad \mathscr{D}w(t) = d(\eta)w(t). \tag{13}$$

Satisfying the following assumption:

(A3) $\mathscr{D}$ is bounded, that is, $\langle h, \mathscr{D}h\rangle \leq d\langle h, h\rangle$, where $h \in \mathscr{B}$.

In a similar way, the operator $\mathscr{B}\colon \mathbf{R} \longrightarrow \mathscr{B}^*$ is offered by

$$\langle \mathscr{B}h, g\rangle = \int_0^l b(\eta)h(\eta)g(\eta)\mathrm{d}\eta. \tag{14}$$

with the underlying assumption:

(A4) $\langle h, \mathscr{B}h\rangle \leq \sigma_b\langle h, h\rangle$, that is, $\mathscr{B}$ is bounded.

Then, the distributed parameter system (1) can be rewritten to compact form as follows:

$$\begin{cases} \dot{L}(t) = \mathscr{A}L(t) + \mathscr{D}w(t), \\ z(t) = \mathscr{B}L(t), \end{cases} \tag{15}$$

where $\mathscr{H} = L_2(\Omega)$ is the state space. The current state of the system is $L(t,\cdot) = \{L(t,\eta)\colon 0 \leq \eta \leq l\}$. The Sobolev space $\mathscr{B} = H_0^1(0,l) = \{g \in H^1(\Omega)|g(0) = g(l) = 0\}$ rules over the space $\mathscr{B}$, and its conjugate dual space is $\mathscr{B}^* = H^{-1}(\Omega)$.

Let the infinitesimal operator$\mathscr{A} = (\mathrm{d}/\mathrm{d}\eta)(a(\eta)(\mathrm{d}/\mathrm{d}\eta)) -\varphi$, and its proposed domain is established by $\mathscr{D}(\mathscr{A}) = \{g \in L_2(\Omega)\colon g, g'$ are absolutely continuous, $g'' \in L_2(\Omega)$ and $g(0) = g(l) = 0\}$. The domain $\mathscr{D}(\mathscr{A})$ of the operator $\mathscr{A}$ is dense in $\mathscr{H}$ and thus generates a strongly continuous semigroup $T(t), t \geq 0$ [18].

Remark 3. The following calculation can be employed to verify that the bounded and coercivity assumptions (A1) and (A2) are valid:

$$\begin{aligned} |\langle h, \mathscr{A}g\rangle| &= \left|\int_0^l \left[\frac{\mathrm{d}}{\mathrm{d}\eta}\left(a(\eta)\frac{\mathrm{d}h(\eta)}{\mathrm{d}\eta}\right) - \varphi(h(\eta))h(\eta)\right]g(\eta)\mathrm{d}\eta\right| \leq a_0\left|\int_0^l \frac{\mathrm{d}h(\eta)}{\mathrm{d}\eta}\frac{\mathrm{d}g(\eta)}{\mathrm{d}\eta}\mathrm{d}\eta\right| \\ &\quad + \varphi_M\left|\int_0^l h(\eta)g(\eta)\mathrm{d}\eta\right| \\ &\leq a_0\sqrt{\int_0^l \left(\frac{\mathrm{d}h(\eta)}{\mathrm{d}\eta}\right)^2\mathrm{d}\eta}\sqrt{\int_0^l \left(\frac{\mathrm{d}g(\eta)}{\mathrm{d}\eta}\right)^2\mathrm{d}\eta} \\ &\quad + \varphi_M\sqrt{\int_0^l h^2(\eta)\mathrm{d}\eta}\sqrt{\int_0^l g^2(\eta)\mathrm{d}\eta} \\ &= a_0\|h\|\|g\| + \varphi_M|h||g| \\ &\leq a_0\|h\|\|g\| + \varphi_M\alpha^2\|h\|\|g\| \\ &= \varrho_0\|h\|\|g\|, \end{aligned} \tag{16}$$

where $\varrho_0 = a_0 + \varphi_M\alpha^2$. The bounded assumption isestablished. For coercivity, you can see the following proof:

$$\begin{aligned} \langle h, -\mathscr{A}h\rangle &= \int_0^l -\left[\frac{\mathrm{d}}{\mathrm{d}\eta}\left(a(\eta)\frac{\mathrm{d}h(\eta)}{\mathrm{d}\eta}\right) - \varphi(h(\eta))h(\eta)\right]h(\eta)\mathrm{d}\eta \\ &\geq a_0\int_0^l \left(\frac{\mathrm{d}h(\eta)}{\mathrm{d}\eta}\right)^2\mathrm{d}\eta + \varphi_m\int_0^l h^2(\eta)\mathrm{d}\eta \\ &= a_0\|h\|^2 + \varphi_m|h|^2 \\ &\geq a_0\|h\|^2. \end{aligned} \tag{17}$$

In the same way, the measurement output (4) is also expressed as

$$y(t) = \Lambda_\beta(t)\mathscr{C}(\eta^s(t))L(t), \tag{18}$$

where $\Lambda_\beta(t) = \mathrm{diag}\{\beta_1(t), \beta_2(t), \dots, \beta_n(t)\}$ and $y(t) = [y_1(t), y_2(t), \dots, y_n(t)]^T$. The operator $\mathscr{C}(\eta^s(t))$: $\mathscr{B} \longrightarrow \underbrace{\mathbf{R}\times\mathbf{R}\times\cdots\times\mathbf{R}}_{n}$ which described the output is given by

$$\langle \mathscr{C}(\eta^s(t))h, g\rangle = \begin{bmatrix} \int_0^l c_1(\eta; \eta_1^s(t))h(\eta)g(\eta)\mathrm{d}\eta \\ \int_0^l c_2(\eta; \eta_2^s(t))h(\eta)g(\eta)\mathrm{d}\eta \\ \vdots \\ \int_0^l c_n(\eta; \eta_n^s(t))h(\eta)g(\eta)\mathrm{d}\eta \end{bmatrix}, \tag{19}$$

where the vector of the sensor time-varying location function as a parameter is then expressed as $\eta^s(t) = [\eta_1^s(t), \eta_2^s(t), \dots, \eta_n^s(t)]^T$. In fact, as nonnegative distribution $c_i(\eta; \eta_i^s(t))(i = 1, 2, \dots, n)$ is bounded, $\mathscr{C}(\eta^s(t))$ satisfies the following assumption:

(A5) $\mathscr{C}(\eta^s(t))$ is bounded, that is, $\langle h, \mathscr{C}(\eta^s(t))h\rangle \le \sigma_c\langle h, h\rangle$.

One can directly see that the observation operator is self-adjoint and that its norm is determined by the embedding constant α and the measure of the spatial domain $\mu(\Omega)$.

In this paper, the state of the target (1) is estimated by taking into account a mobile sensor network with multiple missing data. Here, the following filter structure for mobile sensors i is given:

$$\begin{cases} \dot{\widehat{L}}_i(t) = \mathscr{A}\widehat{L}_i(t) + \mathscr{C}^*(\eta_i^s(t))\gamma_i\left[y_i(t) - \overline{\beta}_i\mathscr{C}(\eta_i^s(t))\widehat{L}_i(t)\right] - G_i\sum_{j\neq i}\left(\widehat{L}_i(t) - \widehat{L}_j(t)\right), \\ \widetilde{z}_i(t) = \mathscr{B}\widehat{L}_i(t), \end{cases} \tag{20}$$

where $\widehat{L}_i(t)$ denotes the state estimation of ith mobile sensor, $\widehat{z}_i(t)$ denotes the estimation of $z(t)$ after applying the filter, $y_i(t)$ is the output measurement of the ith moving sensing device, that is, $y_i(t) = \beta_i(t)\mathscr{C}(\eta_i^s(t))L(t)$, and $\mathscr{C}^*(\eta_i^s(t))$ is the adjoint of the observation operator $\mathscr{C}(\eta_i^s(t))$. Observer gains are denoted by $\gamma_i > 0$, whereas consensus filter gains are denoted by G_i. Furthermore, $\widehat{L}_i(0) = \widehat{L}_{i0} \neq L(0)$for all $i = 1, 2, \dots, n$.

Letting $e_i(t) = L(t) - \widehat{L}_i(t)$ and $\widetilde{z}_i(t) = z(t) - \widehat{z}_i(t)$, (15) and (20) can be employed to derive the filtering error system as follows:

$$\begin{cases} \dot{e}_i(t) = \mathscr{A}_c(\eta_i^s(t))e_i(t) - \mathscr{C}^*(\eta_i^s(t))\gamma_i\left(\beta_i(t) - \overline{\beta}_i\right)\mathscr{C}(\eta_i^s(t))\eta(t) \\ +G_i\sum_{j\neq i}\left(e_j(t) - e_i(t)\right) + \mathscr{D}w(t), \\ \widetilde{z}_i(t) = \mathscr{B}e_i(t), \end{cases} \tag{21}$$

where $\mathscr{A}_c(\eta_i^s(t)) = \mathscr{A} - \mathscr{C}^*(\eta_i^s(t))\gamma_i\overline{\beta}_i\mathscr{C}(\eta_i^s(t))$, also with $e_i(0) = L(0) - \widehat{L}_i(0) \neq 0$.

Given the fact that $\mathscr{C}(\eta_i^s(t))$ is self-adjoint, it implies that the closed-loop operator $\mathscr{A}_c(\eta_i^s(t))$ is self-adjoint. Also, $\mathscr{A}_c(\eta_i^s(t))$ is invertible when (A1), (A2), and (A5) are combined.

Our goal in solving the issue raised is to build a distributed H_∞ consensus filter that takes the form of (20) so as to achieve the following conditions:

(1) The filtering error system with $w(t) = 0$ has a zero solution that is globally asymptotically stable in the mean square.

(2) For the provided disturbance attenuation level $\gamma > 0$, under the zero initial condition, the H_∞ consensus performance constraint fits the following inequality:

$$\frac{1}{n}\sum_{i=1}^n \left|\widetilde{z}_i(t)\right|^2 \le \gamma^2|w(t)|^2. \tag{22}$$

For the purpose of obtaining meaningful results, the definition and lemmas stated below have been introduced.

Definition 1. The filtering error system (21) with $w(t) = 0$ is said to be globally asymptotically stable in the mean square if

$$\lim_{t\longrightarrow+\infty}\mathbb{E}\left|e_i(t)\right|^2 = 0, \tag{23}$$

which holds for any $i \in \{1, 2, \dots, n\}$.

Definition 2. The filters (20) are said to be distributed H_∞ consensus filters if their filtering error $\widetilde{z}_i(t)$ satisfy the following inequalities:

$$\frac{1}{n}\sum_{i=1}^n \left|\widetilde{z}_i(t)\right|^2 \le \gamma^2|w(t)|^2, \tag{24}$$

where the disturbance attenuation level $\gamma > 0$ is given, for any $i \in \{1, 2, \dots, n\}$.

Remark 4. The average filtering error ought to satisfy the H_∞ performance constraint in the sensor network when the value of the filtering error $\widetilde{z}_i(t), i = 1, 2, \dots, n$ satisfies the H_∞ consensus performance constraint. Moreover, the H_∞

consensus performance constraint will reduce the H_∞ performance constraint if only one sensor exists in the network.

Lemma 1. *Assuming that ϵ be a positive scalar and that v_1, v_2 be any n-dimensional real vector. The following inequality is thus true:*

$$2\langle v_1, v_2\rangle \le \epsilon\langle v_1, v_1\rangle + \epsilon^{-1}\langle v_2, v_2\rangle. \tag{25}$$

Lemma 2. *(Barbalat's lemma [19]). In the scenario that a nonnegative function $f(t)$ is the Lebesgue integer and uniformly continuous on $[0, +\infty)$, then $\lim_{t\longrightarrow+\infty} f(t) = 0$.*

The corresponding theorem contains the primary results of this work.

3. Main Results and Proofs

3.1. Stability Analysis

Theorem 1. *The spatial distribution of the mobile sensors (12) and the consensus filter (20) are given. If there exist two positive constants p_i and q_i such that the following inequalities hold:*

$$\begin{aligned} &q_i \ge \frac{1}{2},\\ &p_i \ge \frac{\gamma_i^2\overline{\beta}_i\left(1-\overline{\beta}_i\right)\sigma_c^4\alpha^2}{4a_0}, \end{aligned} \tag{26}$$

and the following is the mobile sensor velocity law:

$$\dot{\eta}_i^s(t) = -\rho_i\gamma_i\overline{\beta}_i\sum_{j=1}^{n} W_{ij}\left[1+\kappa_i\sum_{k\ne i}\left(\sum_{j=1}^{n} W_{kj}\right)^2\right], \tag{27}$$

where

$$\begin{aligned} W_{ij} = &\int_{\eta_{i0}^s+(j-1)\Delta h}^{\eta_{i0}^s+j\Delta h} \frac{\partial c_{ij}(\eta)}{\partial\eta} c_{ij}(\eta)e_i^2(t,\eta)\mathrm{d}\eta + c_{ij}^2\left(\eta_{i0}^s+(j-1)\Delta h+0\right)e_i^2\left(t,\eta_{i0}^s+(j-1)\Delta h\right)\\ &- c_{ij}^2\left(\eta_{i0}^s+j\Delta h-0\right)e_i^2\left(t,\eta_{i0}^s+j\Delta h\right), i, j, k = 1, 2, \ldots, n, \end{aligned} \tag{28}$$

with $\Delta h = \varepsilon^+ + \varepsilon^-/m$, $\eta_{i0}^s = \eta_i^s - \varepsilon^-$, and $\rho_i > 0$ denotes the velocity gain of ith mobile sensor; the zero solution of a filtering error system (21) with $w(t) = 0$ is globally asymptotically stable in the mean square. The mobile sensing scheme improves the filter performance by accelerating the convergence of the filtering error $e_i(t)$ to zero.

Proof. It is simple to verify the closed-loop operator $\mathscr{A}_c(\eta_i^s(t))$ from (A1) and (A2), which satisfies the following criteria.

$$\begin{aligned} \left|\sum_{i=1}^{n}\langle h, \mathscr{A}_c(\eta_i^s(t))g\rangle\right| &= \left|\sum_{i=1}^{n}\langle h, \left(\mathscr{A} - \mathscr{C}^*(\eta_i^s(t))\gamma_i\overline{\beta}_i\mathscr{C}(\eta_i^s(t))\right)g\rangle\right| \le \varrho_0 n\|h\|\|g\|\\ &\quad + \langle h, \mathscr{C}^*(\eta^s(t))\Gamma\overline{\Lambda}_\beta\mathscr{C}(\eta^s(t))g\rangle\\ &= \varrho_0 n\|h\|\|g\| + \langle\overline{\Lambda}_\beta\Gamma\mathscr{C}(\eta^s(t))h, \mathscr{C}(\eta^s(t))g\rangle\\ &\le \varrho_0 n\|h\|\|g\| + \lambda_{\max}\left(\overline{\Lambda}_\beta\Gamma\right)\alpha^2\mu^2(\Omega)\|h\|\|g\|\\ &= \varrho\|h\|\|g\|, \end{aligned} \tag{29}$$

where $\varrho = \varrho_0 n + \lambda_{\max}(\overline{\Lambda}_\beta\Gamma)\alpha^2\mu^2(\Omega) > 0$, $\overline{\Lambda}_\beta = \mathrm{diag}\{\overline{\beta}_1, \overline{\beta}_2, \ldots, \overline{\beta}_n\}$, $\Gamma = \mathrm{diag}\{\gamma_1, \gamma_2, \ldots, \gamma_n\}$, and $\|\mathscr{C}(\eta^s(t))\| = \alpha\mu(\Omega)$.

$$\begin{aligned} \sum_{i=1}^{n}\langle h, -\mathscr{A}_c(\eta_i^s(t))h\rangle &= \sum_{i=1}^{n}\langle h, -\left(\mathscr{A} - \mathscr{C}^*(\eta_i^s(t))\gamma_i\overline{\beta}_i\mathscr{C}(\eta_i^s(t))\right)h\rangle \ge a_0 n\|h\|^2\\ &\quad + \langle h, \mathscr{C}^*(\eta^s(t))\Gamma\overline{\Lambda}_\beta\mathscr{C}(\eta^s(t))h\rangle = a_0 n\|h\|^2\\ &\quad + \langle\overline{\Lambda}_\beta\Gamma\mathscr{C}(\eta^s(t))h, \mathscr{C}(\eta^s(t))h\rangle \ge a_0 n\|h\|^2\\ &\quad + \lambda_{\min}\left(\overline{\Lambda}_\beta\Gamma\right)\left|\mathscr{C}(\eta^s(t))h\right|^2 > a_0 n\|h\|^2. \end{aligned} \tag{30}$$

Considering the following parameter-dependent Lyapunov functional, we obtain

$$V(t) = -\sum_{i=1}^{n}\langle e_i(t), \mathscr{A}_c(\eta_i^s(t))e_i(t)\rangle + \sum_{i=1}^{n}\langle L(t), p_i L(t)\rangle. \tag{31}$$

As defined by $\mathscr{L}V(t) = \lim_{\Delta\longrightarrow 0^+}(\mathrm{E}\{V(t+\Delta)|t\} - V(t))/\Delta$, by applying the dynamics of the filteringerror (21), we obtain $\mathscr{L}V$ that

$$\begin{aligned}\mathscr{L}V(t) = &-\sum_{i=1}^{n}\mathbb{E}\langle \dot{e}_i(t), \mathscr{A}_c(\eta_i^s(t))e_i(t)\rangle - \sum_{i=1}^{n}\mathbb{E}\langle e_i(t), \mathscr{A}_c(\eta_i^s(t))\dot{e}_i(t)\rangle\\ &-\sum_{i=1}^{n}\mathbb{E}\langle e_i(t), \frac{\mathrm{d}\mathscr{A}_c(\eta_i^s(t))}{\mathrm{d}t}e_i(t)\rangle\\ &+\sum_{i=1}^{n}\langle \dot{L}(t), p_i\eta(t)\rangle\\ &+\sum_{i=1}^{n}\langle \eta(t), p_i\dot{L}(t)\rangle.\end{aligned} \tag{32}$$

It was simple to arrive at the following conclusion by taking into account (6) and (7) and noting that $\mathscr{A}_c(\eta_i^s(t))$ is self-adjoint.

$$\begin{aligned}&-\sum_{i=1}^{n}\mathbb{E}\langle \dot{e}_i(t), \mathscr{A}_c(\eta_i^s(t))e_i(t)\rangle - \sum_{i=1}^{n}\mathbb{E}\langle e_i(t), \mathscr{A}_c(\eta_i^s(t))\dot{e}_i(t)\rangle\\ &= -2\sum_{i=1}^{n}\mathbb{E}\left\langle \begin{array}{c}\mathscr{A}_c(\eta_i^s(t))e_i(t) - \mathscr{C}^*(\eta_i^s(t))\gamma_i(\beta_i(t)-\overline{\beta}_i)\mathscr{C}(\eta_i^s(t))L(t)\\ +G_i\sum_{k\neq i}(e_k(t)-e_i(t)), \mathscr{A}_c(\eta_i^s(t))e_i(t)\end{array}\right\rangle\\ &= -2\sum_{i=1}^{n}\mathbb{E}\langle \mathscr{A}_c(\eta_i^s(t))e_i(t), \mathscr{A}_c(\eta_i^s(t))e_i(t)\rangle\\ &\quad + 2\sum_{i=1}^{n}\mathbb{E}\langle \mathscr{C}^*(\eta_i^s(t))\gamma_i(\beta_i(t)-\overline{\beta}_i)\mathscr{C}(\eta_i^s(t))L(t), \mathscr{A}_c(\eta_i^s(t))e_i(t)\rangle\\ &\quad - 2\sum_{i=1}^{n}\mathbb{E}\langle G_i\sum_{k\neq i}(e_k(t)-e_i(t)), \mathscr{A}_c(\eta_i^s(t))e_i(t)\rangle.\end{aligned} \tag{33}$$

Observing the assumption (A5) and Lemma 1, the following is true:

$$\begin{aligned}&2\sum_{i=1}^{n}\mathbb{E}\langle \mathscr{C}^*(\eta_i^s(t))\gamma_i(\beta_i(t)-\overline{\beta}_i)\mathscr{C}(\eta_i^s(t))L(t), \mathscr{A}_c(\eta_i^s(t))e_i(t)\rangle\\ &\quad\le \sum_{i=1}^{n}q_i\mathbb{E}\langle \mathscr{C}^*(\eta_i^s(t))\gamma_i(\beta_i(t)-\overline{\beta}_i)\mathscr{C}(\eta_i^s(t))L(t), \mathscr{C}^*(\eta_i^s(t))\gamma_i(\beta_i(t)-\overline{\beta}_i)\mathscr{C}(\eta_i^s(t))L(t)\rangle\\ &\qquad + \sum_{i=1}^{n}q_i^{-1}\mathbb{E}\langle \mathscr{A}_c(\eta_i^s(t))e_i(t), \mathscr{A}_c(\eta_i^s(t))e_i(t)\rangle\end{aligned}$$

$$\begin{aligned}
&= \sum_{i=1}^{n} q_i \gamma_i^2 \overline{\beta}_i \left(1-\overline{\beta}_i\right) \left|\mathscr{C}^*\left(\eta^s(t)\right)\mathscr{C}\left(\eta^s(t)\right)L(t)\right|^2 \\
&\quad + \sum_{i=1}^{n} q_i^{-1} \mathbb{E}\langle \mathscr{A}_c\left(\eta_i^s(t)\right)e_i(t), \mathscr{A}_c\left(\eta_i^s(t)\right)e_i(t)\rangle \\
&\le \sum_{i=1}^{n} q_i \gamma_i^2 \overline{\beta}_i \left(1-\overline{\beta}_i\right)\sigma_c^4 |L(t)|^2 \\
&\quad + \sum_{i=1}^{n} q_i^{-1} \mathbb{E}\langle \mathscr{A}_c\left(\eta_i^s(t)\right)e_i(t), \mathscr{A}_c\left(\eta_i^s(t)\right)e_i(t)\rangle,
\end{aligned} \tag{34}$$

for any scalar $q_i > 0\,(i = 1, 2, \ldots, n)$.

Choosing the consensus filter here results in $G_i = \mathscr{A}_c^{-1}\left(\eta_i^s(t)\right)$ for simplicity, and we have

$$\begin{aligned}
&-2\sum_{i=1}^{n} \mathbb{E}\langle G_i \sum_{k\ne i}\left(e_k(t)-e_i(t)\right), \mathscr{A}_c\left(\eta_i^s(t)\right)e_i(t)\rangle \\
&= -2\sum_{i=1}^{n} \langle \sum_{k\ne i}\left(e_k(t)-e_i(t)\right), e_i(t)\rangle \\
&= -2\sum_{k\ne i} \langle e_k(t)-e_i(t), e_k(t)-e_i(t)\rangle \\
&= -2\sum_{k\ne i} \left|e_k(t)-e_i(t)\right|^2.
\end{aligned} \tag{35}$$

Substituting (34)-(35) into (36) leads to

$$\begin{aligned}
&-\sum_{i=1}^{n} \mathbb{E}\langle \dot{e}_i(t), \mathscr{A}_c\left(\eta_i^s(t)\right)e_i(t)\rangle - \sum_{i=1}^{n} \mathbb{E}\langle e_i(t), \mathscr{A}_c\left(\eta_i^s(t)\right)\dot{e}_i(t)\rangle \\
&= \sum_{i=1}^{n}\left(-2+q_i^{-1}\right)\mathbb{E}\left|\mathscr{A}_c\left(\eta_i^s(t)\right)e_i(t)\right|^2 \\
&\quad + \sum_{i=1}^{n} q_i \gamma_i^2 \overline{\beta}_i\left(1-\overline{\beta}_i\right)\sigma_c^4 |L(t)|^2 - 2\sum_{k\ne i}\left|e_k(t)-e_i(t)\right|^2.
\end{aligned} \tag{36}$$

Then, the third term of (32) has

$$\begin{aligned}
&-\sum_{i=1}^{n} \mathbb{E}\langle e_i(t), \frac{\mathrm{d}\mathscr{A}_c\left(\eta_i^s(t)\right)}{\mathrm{d}t} e_i(t)\rangle \\
&= \sum_{i=1}^{n} \mathbb{E}\langle e_i(t), \frac{\mathrm{d}}{\mathrm{d}t}\left(\mathscr{C}^*\left(\eta_i^s(t)\right)\gamma_i\overline{\beta}_i\mathscr{C}\left(\eta_i^s(t)\right)\right)e_i(t)\rangle \\
&= 2\sum_{i=1}^{n} \gamma_i\overline{\beta}_i \langle \mathscr{C}\left(\eta_i^s(t)\right)e_i(t), \dot{\eta}_i^s(t)\frac{\mathrm{d}\mathscr{C}\left(\eta_i^s(t)\right)}{\mathrm{d}\eta} e_i(t)\rangle \\
&= 2\sum_{i=1}^{n} \dot{\eta}_i^s(t)\gamma_i\overline{\beta}_i \int_0^l c_i\left(\eta;\eta_i^s(t)\right)\frac{\partial c_i\left(\eta;\eta_i^s(t)\right)}{\partial\eta} e_i^2(t,\eta)\mathrm{d}\eta \\
&= 2\sum_{i=1}^{n} \dot{\eta}_i^s(t)\gamma_i\overline{\beta}_i \sum_{j=1}^{n}\int_{\eta_{i0}^s+(j-1)\Delta h}^{\eta_{i0}^s+j\Delta h} c_{ij}(\eta)\frac{\partial}{\partial\eta}\left[c_{ij}(\eta)\left(H\left(\eta-\left(\eta_{i0}^s+(j-1)\Delta h\right)\right)-H\left(\eta-\left(\eta_{i0}^s+j\Delta h\right)\right)\right)\right]e_i^2(t,\eta)\mathrm{d}\eta \\
&= 2\sum_{i=1}^{n} \dot{\eta}_i^s(t)\gamma_i\overline{\beta}_i \sum_{j=1}^{n}\int_{\eta_{i0}^s+(j-1)\Delta h}^{\eta_{i0}^s+j\Delta h} \begin{bmatrix} \dfrac{\partial c_{ij}(\eta)}{\partial\eta}\left(H\left(\eta-\left(\eta_{i0}^s+(j-1)\Delta h\right)\right)-H\left(\eta-\left(\eta_{i0}^s+j\Delta h\right)\right)\right) \\ +c_{ij}(\eta)\left(\delta\left(\eta-\left(\eta_{i0}^s+(j-1)\Delta h\right)\right)-\delta\left(\eta-\left(\eta_{i0}^s+j\Delta h\right)\right)\right) \end{bmatrix} c_{ij}(\eta)e_i^2(t,\eta)\mathrm{d}\eta \\
&= 2\sum_{i=1}^{n} \dot{\eta}_i^s(t)\gamma_i\overline{\beta}_i \sum_{j=1}^{n} W_{ij}.
\end{aligned} \tag{37}$$

Here, $W_{ij}, i, j = 1, 2, \ldots, n$ are defined in Theorem 1.

The selection

$$\dot{\eta}_i^s(t) = -\rho_i \gamma_i \overline{\beta}_i \sum_{j=1}^{n} W_{ij}, \quad (38)$$

yields in (37) negative definite, where $\rho_i > 0\,(i = 1, 2, \ldots, n)$ is the diagonal entries of the positive definite diagonal matrix Λ_ρ. It is worth noting that $\dot{\eta}_i^s(t)$ indicates the velocity of each mobile sensing device.

Moreover, the choice of the velocity law for the mobile sensing device is decoupled, meaning that it is only taken into account for its own measurement. And in a real mobile sensor network, not only its velocity depends on itself, but also its neighbors contribute to its velocity law. Every mobile sensor (38), regardless of its velocity law, can be modified to

$$\dot{\eta}_i^s(t) = -\rho_i \gamma_i \overline{\beta}_i \sum_{j=1}^{n} W_{ij} \left[1 + \kappa_i \sum_{k \neq i} \left(\sum_{j=1}^{n} W_{kj} \right)^2 \right], \quad (39)$$

where $\kappa_i > 0, i, j, k = 1, 2, \ldots, n$.

Calculating the derivative of the second part of the Lyapunov function yields

$$\begin{aligned} &\sum_{i=1}^{n} \langle \dot{L}(t), p_i \eta(t) \rangle + \sum_{i=1}^{n} \langle \eta(t), p_i \dot{L}(t) \rangle \\ &= 2 \sum_{i=1}^{n} p_i \langle L(t), \mathscr{A} L(t) \rangle \\ &\leq -2 \sum_{i=1}^{n} p_i a_0 \|L(t)\|^2 \leq -2 \sum_{i=1}^{n} \frac{p_i a_0}{\alpha^2} |L(t)|^2. \end{aligned} \quad (40)$$

By substituting (36), (39)–(40) into (32), we obtain

$$\begin{aligned} \mathscr{L}V(t) \leq & \sum_{i=1}^{n} \left(-2 + q_i^{-1} \right) \mathbb{E} \left| \mathscr{A}_c(\eta_i^s(t)) e_i(t) \right|^2 + \sum_{i=1}^{n} \left(q_i \gamma_i^2 \overline{\beta}_i \left(1 - \overline{\beta}_i \right) \sigma_c^4 - 2 \frac{p_i a_0}{\alpha^2} \right) |L(t)|^2 \\ & -2 \sum_{k \neq i} \mathbb{E} \left| e_k(t) - e_i(t) \right|^2 - 2 \sum_{i=1}^{n} \rho_i \gamma_i^2 \overline{\beta}_i^2 \left(\sum_{j=1}^{n} W_{ij} \right)^2 \left[1 + \kappa_i \sum_{k \neq i} \left(\sum_{j=1}^{n} W_{kj} \right)^2 \right]. \end{aligned} \quad (41)$$

The inequalities (26) are feasible, which implies $\mathscr{L}V(t) \leq 0$.

Thus, it is easy to deduce from (41) and the embedding that

$$\begin{aligned} \mathscr{L}V(t) &\leq -\sum_{i=1}^{n} \mathbb{E} \left| \mathscr{A}_c(\eta_i^s(t)) e_i(t) \right|^2 \\ &\leq -\alpha_0 \sum_{i=1}^{n} \mathbb{E} \left| e_i(t) \right|^2, \end{aligned} \quad (42)$$

where $\alpha_0 > 0$ is a constant that contains the embedding constant α and the coercivity constant a_0. By employing the Itô formula, combined with (42), we can obtain

$$\begin{aligned} \mathbb{E}V(t) &= \mathbb{E}V(0) + \int_0^t \mathscr{L}V(s) \mathrm{d}s \\ &\leq \mathbb{E}V(0) - \alpha_0 \int_0^t \sum_{i=1}^{n} \mathbb{E} \left| e_i(t) \right|^2 \mathrm{d}s, \end{aligned} \quad (43)$$

which implies that

$$\int_0^t \sum_{i=1}^{n} \mathbb{E} \left| e_i(t) \right|^2 \mathrm{d}s \leq \frac{1}{\alpha_0} V(0). \quad (44)$$

In addition, it can confirm that $\sum_{i=1}^{n} \mathbb{E}|e_i(t)|^2$ is uniformly continuous on $[0, +\infty)$. Consequently, it follows from Lemma 1 that

$$\lim_{t \longrightarrow +\infty} \mathbb{E}|e(t)|^2 = 0. \quad (45)$$

The proof is complete if the filtering error system (21) with $w(t) = 0$ is globally asymptotically stable in the mean square, as stated in Definition 1. □

Remark 5. In theory, if we select the Lyapunov functional as

$$V(t) = -e^{rt} \sum_{i=1}^{n} \langle e_i(t), \mathscr{A}_c(\eta_i^s(t)) e_i(t) \rangle + e^{rt} \sum_{i=1}^{n} \langle L(t), p_i L(t) \rangle, \quad (46)$$

then the filtering error system (21) with $w(t) = 0$ can also be proved to be globally exponentially stable in the mean square as the similar way in Theorem 1.

3.2. H_∞ Consensus Performance Analysis. Next, we focus on analyzing the H_∞ performance of the filtering error system (21) under the zero initial condition.

Theorem 2. *The disturbance attenuation level $\gamma > 0$ and the filter parameter γ_i and G_i are given. Under the assumption (A1)–(A5), the zero solution of the filtering error system (21) with $w(t) = 0$ is globally asymptotically stable in the mean square. The H_∞ consensus performance (22) is achieved for all nonzero $w(t)$ if, with the initial condition, such that the following matrix inequality holds:*

$$\Psi = \begin{bmatrix} -\alpha_0 + \sigma_b^2 & d\varrho \\ d\varrho & -\gamma^2 \end{bmatrix} < 0. \tag{47}$$

Proof. (42) implies $\Psi < 0$, which is simple to prove. In consideration of Theorem 1, the filtering error system (21) is globally asymptotically stable in the mean square. Let us now concentrate on how the closed-loop system performs in terms of the H_∞ consensus. Likewise, as in Theorem 1, we build the Lyapunov functional candidate $V(t)$. The same line computation used in Theorem 1 results in

$$\begin{aligned} \mathscr{L}V(t) &\leq -\alpha_0 \sum_{i=1}^{n} \mathbb{E}\left|e_i(t)\right|^2 + 2\sum_{i=1}^{n} \mathbb{E}\langle \mathscr{D}w(t), -\mathscr{A}_c(\eta_i^s(t))e_i(t)\rangle \\ &\leq \sum_{i=1}^{n} \mathbb{E}\langle \zeta(t), \widehat{\Psi}\zeta(t)\rangle, \end{aligned} \tag{48}$$

where $\zeta(t) = [e(t), w(t)]^T$ and $\widehat{\Psi} = \begin{bmatrix} -\alpha_0 & d\varrho \\ d\varrho & 0 \end{bmatrix}$.

In order to address the H_∞ consensus performance of the system (21), we provide

$$\begin{aligned} J &= \mathbb{E}\sum_{i=1}^{n} \int_0^T \left|\tilde{z}_i(t)\right|^2 - \gamma^2 |w(t)|^2 \mathrm{d}t \\ &= \mathbb{E}\sum_{i=1}^{n} \int_0^T \langle \tilde{z}_i(t), \tilde{z}_i(t)\rangle \mathrm{d}t - n\gamma^2 \int_0^T \langle w(t), w(t)\rangle \mathrm{d}t. \end{aligned} \tag{49}$$

For any nonzero external disturbances $w(t)$ from (47) to (48), under the zero initial condition, we obtain

$$\begin{aligned} J &\leq \mathbb{E}\sum_{i=1}^{n} \int_0^{T_f} \langle \tilde{z}_i(t), \tilde{z}_i(t)\rangle - \gamma^2 \langle w(t), w(t)\rangle + \mathscr{L}V(t)\mathrm{d}t \\ &\leq \mathbb{E}\sum_{i=1}^{n} \int_0^{T_f} \langle \mathscr{B}e_i(t), \mathscr{B}e_i(t)\rangle - \gamma^2 \langle w(t), w(t)\rangle + \sum_{i=1}^{n} \mathbb{E}\langle \zeta(t), \widehat{\Psi}\zeta(t)\rangle \mathrm{d}t \\ &\leq \sum_{i=1}^{n} \int_0^{T_f} \mathbb{E}\langle \zeta(t), \Psi\zeta(t)\rangle \mathrm{d}t, \end{aligned} \tag{50}$$

where Ψ is defined in Theorem 2.

Along the same lines as in the argument of Theorem 1, we can establish that $J < 0$. Letting $T_f \longrightarrow \infty$, we have

$$\sum_{i=1}^{n} \int_0^T \langle \tilde{z}_i(t), \tilde{z}_i(t)\rangle \mathrm{d}t < n\gamma^2 \int_0^T \langle w(t), w(t)\rangle \mathrm{d}t, \tag{51}$$

namely,

$$\frac{1}{n}\sum_{i=1}^{n} \left|\tilde{z}_i(t)\right|^2 < \gamma^2 |w(t)|^2. \tag{52}$$

The theorem's proof here is completed. □

4. Some Special Case Analysis

Owing to the generality of the result stated in Theorem 1, we now discuss its few useful applications in the following section.

The argument for following Theorem 3 is omitted since it can be inferred from Theorem 1, Theorem 2, and the spatial distribution of moving sensors (8).

Theorem 3. *The disturbance attenuation level $\gamma > 0$ and the filter parameter γ_i are given. Under assumptions (A1)–(A5) and inequalities (26), for the distributed parameter system (15), an H_∞ consensus filter (20) is then constructed so that the filtering error system (21) with $w(t) = 0$ is globally asymptotically stable in the mean square and also satisfies (22) under the zero initial condition for all non-zero $w(t)$ if the mobile sensing device has a spatial distribution as (8), and its velocity law is as follows:*

$$\dot{\eta}_i^s(t) = -\rho_i \gamma_i \overline{\beta}_i W_i \left(1 + \kappa_i \sum_{j \neq i} W_j^2\right), \tag{53}$$

where

$$W_i = \int_{\eta_i^s - \varepsilon_i}^{\eta_i^s + \varepsilon_i} \frac{\partial c_i(\eta)}{\partial \eta} c_i(\eta) e_i^2(t,\eta)\mathrm{d}\eta + c_i^2(\eta_i^s - \varepsilon_i + 0)e_i^2(t,\eta_i^s - \varepsilon_i) - c_i^2(\eta_i^s + \varepsilon_i - 0)e_i^2(t,\eta_i^s + \varepsilon_i), i, j = 1, 2, \ldots, n, \tag{54}$$

where $\rho_i > 0$ denotes each sensor's velocity gain, and the matrix inequality (47) holds.

Hereafter, we discuss how to use a homogeneous mobile sensor network to solve the H_∞ filtering issue for the error system (21).

The spatial distribution of a homogeneous mobile sensor with a time-varying location function of $\eta_i^s(t)$ can be given by

$$c(\eta;\eta_i^s) = \begin{cases} \mu, & \text{if } \eta \in [\eta_i^s - \varepsilon, \eta_i^s + \varepsilon], \\ 0, & \text{otherwise.} \end{cases} \tag{55}$$

As Theorem 1 and Theorem 2 easily lead to this theorem, the proof is omitted.

Theorem 4. *The disturbance attenuation level $\gamma > 0$ and the filter parameter γ_i are given. Under assumptions (A1)–(A5) and inequalities (26), for the distributed parameter system (15), an H_∞ consensus filter (20) is then constructed so that the filtering error system (21) with $w(t) = 0$ is globally asymptotically stable in the mean square and also satisfies (22) under the zero initial condition for all nonzero $w(t)$ if the mobile sensing device has a spatial distribution as (55), its velocity law is as follows:*

$$\dot{\eta}_i^s(t) = -\rho_i\gamma_i\overline{\beta}_i\mu^2\left(e_i^2(t,\eta_i^s-\varepsilon) - e_i^2(t,\eta_i^s+\varepsilon)\right)\left(1+\kappa_i\sum_{j\neq i}\left(e_j^2\left(t,\eta_j^s-\varepsilon\right) - e_j^2\left(t,\eta_j^s+\varepsilon\right)\right)^2\right), i, j = 1,2,\ldots,n, \tag{56}$$

where $\rho_i > 0$ denotes each sensor's velocity gain, and the matrix inequality (47) holds.

Remark 6. Actually, for an homogeneous mobile sensor network, by taking $\mu = c(\eta)$, the location distribution of each moving sensor can be generally given as follows:

$$c(\eta;\eta_i^s) = \begin{cases} c(\eta), & \text{if } \eta \in [\eta_i^s - \varepsilon, \eta_i^s + \varepsilon], \\ 0, & \text{otherwise.} \end{cases} \tag{57}$$

In this case, $c(\eta)$ is allowed to be constant or a piecewise smooth function in local.

Remark 7. In this paper, the case of data missing from the sensor network is taken into consideration for the distributed H_∞ consensus filter for the distributed parameter system (1). Specially, let us consider that the measurement output (4) has no randomly occurred missing measurements, i.e., $\beta_i(t) \equiv 1 \in \mathbf{R}, i = 1,2,\ldots,n$, and we choose $\mu = 1$, a common spatial distribution of a mobile sensor, as follows:

$$c(\eta;\eta_i^s) = \begin{cases} 1, & \text{if } \eta \in [\eta_i^s - \varepsilon, \eta_i^s + \varepsilon], \\ 0, & \text{otherwise.} \end{cases} \tag{58}$$

In the light of Theorem 4, each moving sensor's velocity (56) is able to be further simplified as

$$\dot{\eta}_i^s(t) = -\rho_i\gamma_i\left(e_i^2(t,\eta_i^s-\varepsilon) - e_i^2(t,\eta_i^s+\varepsilon)\right)\left(1+\kappa_i\sum_{j\neq i}\left(e_j^2\left(t,\eta_j^s-\varepsilon\right) - e_j^2\left(t,\eta_j^s+\varepsilon\right)\right)^2\right), i = 1,2,\ldots,n, \tag{59}$$

where $\rho_i > 0$ is the velocity gain of each sensing device. This result is the same as the one in [8], which implies that the main results of this paper are extended to the earlier work.

Remark 8. Let us assume that the homogeneous sensor network has all of the sensors fixed. In other words, $\eta_i^s(t) \equiv \eta_i^s(0) := \overline{\eta}_i^s$ is a constant that is independent of time. In this way, the time-spatial operator of the measurement output operator $\mathscr{C}(\eta^s(t))$ is changed to the spatial operator $\mathscr{C}(\overline{\eta}_i^s)$. Since $\mathscr{C}(\overline{\eta}_i^s)$ is a constant operator, we can also express $\mathscr{C}(\overline{\eta}_i^s) := \mathscr{C}$. The measurement output equation (4) can be rewritten as

$$y(t) = \Lambda_\beta(t)\mathscr{C}L(t). \tag{60}$$

For the ith stationary sensor, the suitable filter is

$$\left\{\dot{\widehat{L}}_i(t) = \mathscr{A}\widehat{L}_i(t) + \mathscr{C}^*\gamma_i\left[y_i(t) - \overline{\beta}_i\mathscr{C}\widehat{L}_i(t)\right] - G_i\sum_{j\neq i}\left(\widehat{L}_i(t) - \widehat{L}_j(t)\right), \widehat{z}_i(t) = \mathscr{B}\widehat{L}_i(t),\right. \tag{61}$$

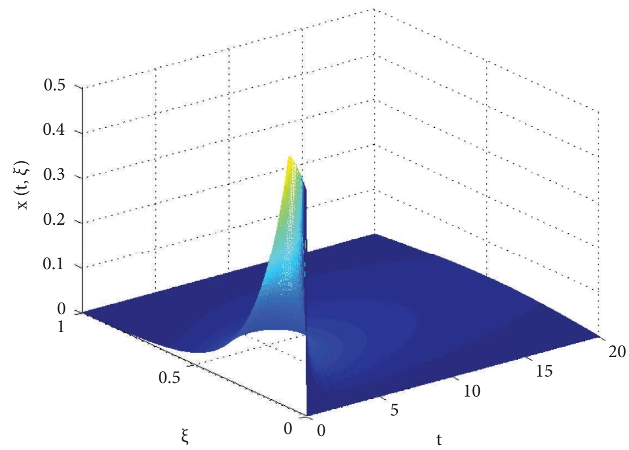

Figure 1: The state evolution of parabolic distributed parameter processes.

where $\widehat{L}_i(0) = \widehat{L}_{i0} \neq L(0)$ for all $i = 1, 2, \ldots, n$. As a result, (15) and (61) can be used to establish the filtering error system in the lines as follows:

$$\begin{cases} \dot{e}_i(t) = \mathscr{A}_c e_i(t) - \mathscr{C}^* \gamma_i \left(\beta_i(t) - \overline{\beta}_i\right) \mathscr{C} L(t) + G_i \sum_{j \neq i} \left(e_j(t) - e_i(t)\right) + \mathscr{D} w(t), \\ \widetilde{z}_i(t) = \mathscr{B} e_i(t), \end{cases} \tag{62}$$

where $\mathscr{A}_c = \mathscr{A} - \gamma_i \overline{\beta}_i \mathscr{C}^* \mathscr{C}$ and $e_i(0) = L(0) - \widehat{L}_i(0) \neq 0$. The following corollary is easily shown in Theorem 3.

Corollary 1. *The disturbance attenuation level $\gamma > 0$ and the filter parameter γ_i and G_i are given. For the distributed parameter system (15), an H_∞ consensus filter (61) is then constructed so that the filtering error system (62) with $w(t) = 0$ is globally asymptotically stable in the mean square and also satisfies (22) for all nonzero $w(t)$ under the zero initial condition if assumptions (A1)–(A5), inequality (26), and the matrix inequality (47) hold.*

5. Numerical Examples

To illustrate the usefulness of the distributed H_∞ consensus filter created in this work, we give a simulated example in this section. The spatial distribution process of temperature in a chemical reactor having Dirichlet boundary conditions and initial conditions is $L(0, \eta) = \sin(\pi\eta) e^{-9\eta^2}, \eta$ in [0, 1]. The evolution of the system is illustrated in Figure 1. The diffusion coefficient is $a_0 = 0.004$. The bounded function is $\varphi(L(t, \eta)) = 1.2 \sin(0.6 L(t, \eta))$. In a kind of spatially distributed process, three mobile sensing devices are considered to collect data and $\eta_1^s(0) = 0.15, \eta_2^s(0) = 0.5$, and $\eta_3^s(0) = 0.85$ are selected for their initial positions.

The spatial distribution of each moving sensing device in $\eta_i^s(t)$ is described by the time-varying location function, which is provided by

$$c(\eta; \eta_i^s) = \begin{cases} 1, & \text{if } \eta \in [\eta_i^s - 0.07, \eta_i^s + 0.07], \\ 0, & \text{otherwise.} \end{cases} \tag{63}$$

The probabilities are considered to be $\overline{\beta}_1 = 0.9, \overline{\beta}_2 = 0.85$, and $\overline{\beta}_3 = 0.8$. As shown in Figure 2, the measurement with missing random data is observed by moving sensors.

Initial conditions are assumed to be $\widehat{L}_1(0, \eta) = \widehat{L}_2(0, \eta) = \widehat{L}_3(0, \eta) = 0$ for the distributed H_∞ consensus filter. The filter gains are given by $\gamma_1 = 80, \gamma_2 = 85$, and $\gamma_3 = 90$. Three distributed H_∞ consensus filters are employed to illustrate the evolution of the filtering error system in Figures 3–5. The filtering error $\widetilde{z}_i(t)$ is given in Figure 6. The output $z(t)$ and associated estimates for mobile sensors are shown in Figure 7. The output $z(t)$ of the ith filter $(i = 1, 2, 3)$ in the moving sensing device and its estimated value are shown in Figure 7.

Three fixed-in-space sensors are considered to be a comparison, which are located at $\eta_1^s = 0.15, \eta_2^s = 0.50$, and $\eta_3^s = 0.85$. The trajectories of the three sensors in stationary and moving scenarios are illustrated in Figure 8.

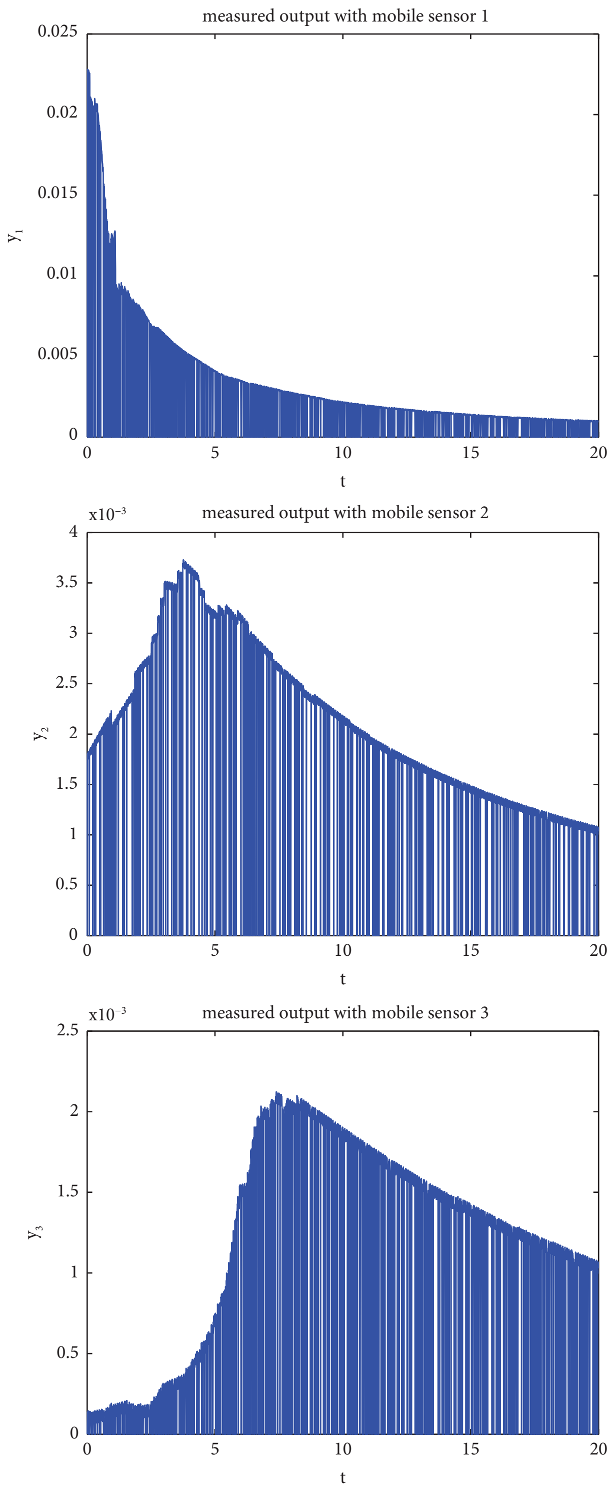

Figure 2: The output of moving sensors with random missing data.

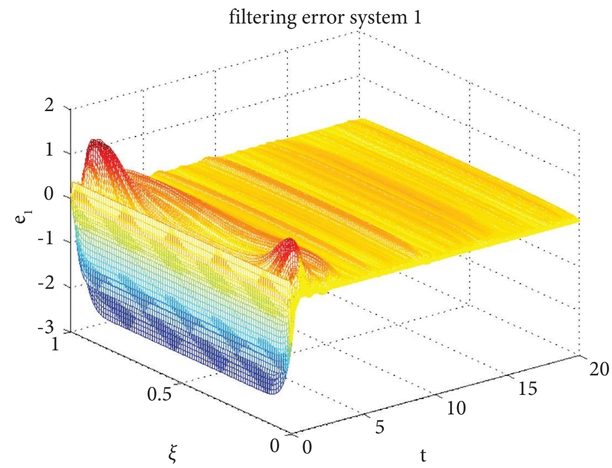

Figure 3: Filter error system development for filter 1.

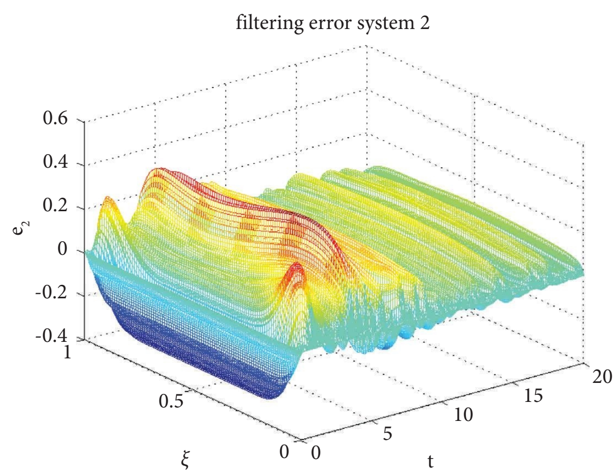

Figure 4: Filter error system development for filter 2.

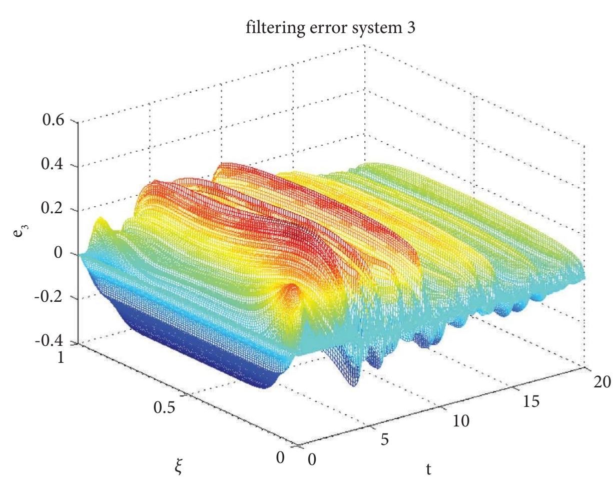

Figure 5: Filter error system development for filter 3.

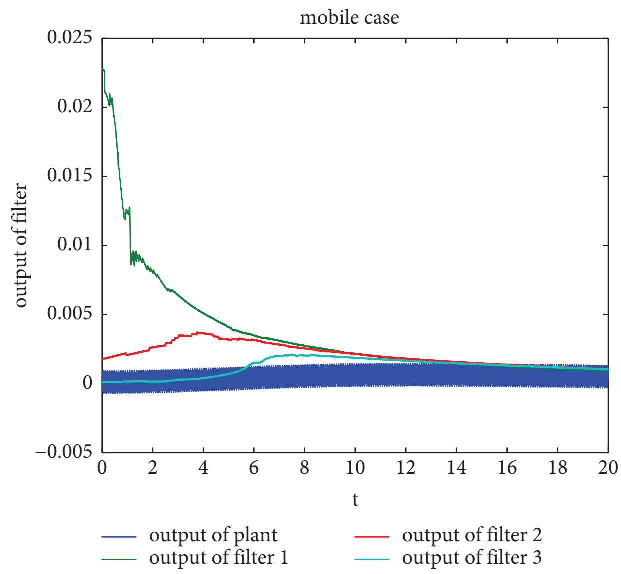

Figure 6: Output of plant and filters.

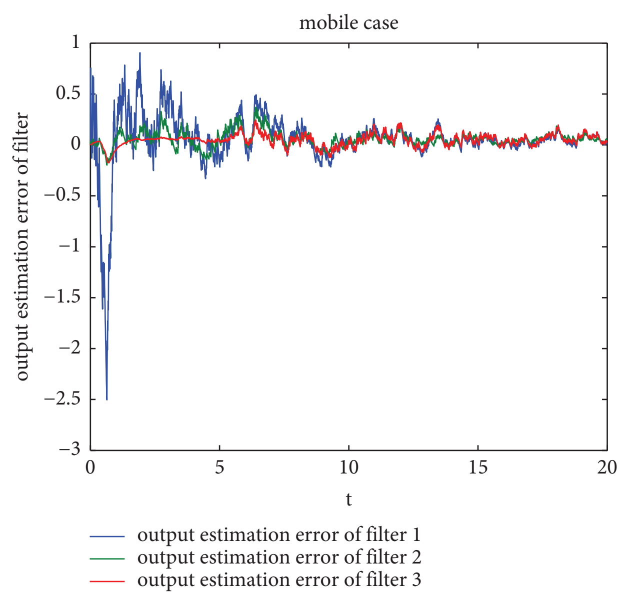

Figure 7: Output estimation errors of filters.

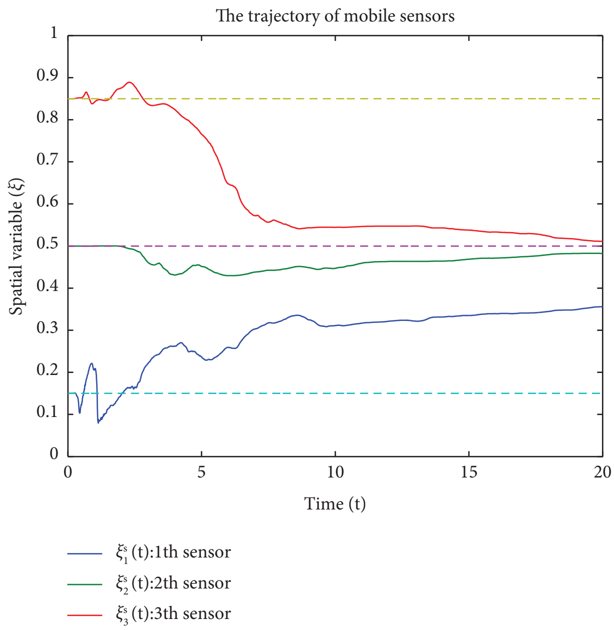

Figure 8: The trajectories of three mobile sensors.

6. Conclusions

The H_∞ distributed consensus filtering issue has been investigated in this paper for an array of parabolic distributed parameter systems with multiple missing measurements. An effective Lyapunov direct approach, which ensures a specific level of H_∞ consensus disturbance rejection attenuation, has been proposed to build the filtering error system that is globally asymptotically stable in the mean square for all permissible randomly occurring missing data. Several sufficient conditions can be obtained within the optimized framework, including the velocity law of the moving sensor, which can ensure a faster convergence of the filtering error to zero. Finally, a numerical simulation is adopted to demonstrate the usefulness of the obtained results of the study.

Data Availability

No underlying data were collected or produced in this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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